Blind Topology Identification for Power Systems

Xiao Li†, H. Vincent Poor* and Anna Scaglione†
†Dept. of Electrical and Computer Engineering, University of California, Davis, CA 95616, USA
*Dept. of Electrical Engineering, Princeton University, NJ 08540, USA

***Abstract*—In this paper, the blind topology identification problem for power systems only using power injection data at each bus is considered. As metering becomes widespread in the smart grid, a natural question arising is how much information about the underlying infrastructure can be inferred from such data. The identifiability of the grid topology is studied, and an efficient learning algorithm to estimate the grid Laplacian matrix (i.e., the graph equivalent of the grid admittance matrix) is proposed. Finally, the performance of our algorithm for the IEEE-14 bus system is demonstrated, and the consistency of the recovered graph with the true graph associated with the underlying power grid is shown in simulations.**

I. Introduction

A major contemporary development associated with the smart grid is the deployment of rapid phasor measurement units (PMUs) in the transmission grid, and the advanced metering infrastructure (AMI) in the distribution system. The value of having the large volume of data generated by the deployment is to provide the basis for informative analytics for various purposes. The most important analytic in the grid is the state of the system, which is usually obtained at the Energy Management System (EMS) by performing state estimation (SE) for monitoring and control [1], [2]. State estimates and other analytics enable better load predictions for market decisions [3], perform contingency analysis, and detect anomalies in the grid (i.e., failures, attacks, etc.). The latter has gained considerable interest recently because the expansion of information systems in the smart grid also has the potential of exposing it to cyber or physical attacks on the grid [4].

A. Challenges and Motivation

The study of malicious attacks is quite recent in power grids [5]. A form of attack that has received considerable attention in the literature consists of injecting bad data into the measurement systems to misguide the results of SE [6]. For instance, there have been extensive studies on designing data attacks on power measurements and topology to pass the bad data test [7]–[9], or influence market operations [10], [11], as well as physical attacks that are unobservable to monitoring devices such as PMUs [12].

Most designs and countermeasures for failures and attacks are based on the assumption that the *topology data* are given, which is not true in many applications (e.g., the distribution system [13]). In transmission networks, topology data are gathered from the field by the Supervisory Control and Data Acquisition (SCADA) system, and sometimes estimated by performing generalized state estimation that includes breaker statuses as variables [1], [14]. Nevertheless, while this information is in general known in transmission networks by the operators, it is more likely that malicious attackers have access to scattered field data instead of the processed data at the EMS, which is supposedly better protected [4]. This is also a perspective that this paper is trying to offer, which is to study whether it is possible for an attacker to learn the topology with accessible but limited data.

Taking the perspective of complex systems, studies have shown that topology data also allows to pinpoint critical nodes via *centrality measures* [15]. It has been shown in [15]–[17] that nodes with high centrality carry substantially higher risk in triggering cascading failures when experiencing disturbances, failures or attacks. In light of this line of research, this work also shows that such measures can be effectively learned.

B. Related Works and Contributions

Existing works on topology estimation are usually focused on estimating or detecting the *changes* that occur in the topology via line outage and fault detection [18]–[20] and generalized state estimation involving circuit breaker statuses [21], and similar issues. Recently, there has been some work on identifying the entire topology under various assumptions [13], [22]. Specifically, [13] assumes that the power injections follow certain probability distributions and uses the voltage correlation to identify the topology in the distribution grid, while [22] uses the transmission time on the power lines to identify the topology in a microgrid environment via hypothesis testing. However, the assumption in [13] on power injections is in general not true and the requirement of being able to use accurate timing information in power line communications [22] and perform complicated hypothesis testing is not easy to implement.

Our contribution in this paper is to cast topology estimation in the framework of blind system identification [23]. We focus specifically on leveraging exclusively on the power injections at different buses without any other data to estimate the weighted graph Laplacian (i.e., analogous to the admittance matrix) associated with the underlying power grid. Although our analysis shows that the Laplacian of a general graph cannot be uniquely identified using solely power injection data, it is also proved in this paper that the eigenvectors of the Laplacian can be uniquely identified. To resolve the ambiguity in the eigenvalues of the Laplacian, we propose to use the sparsity of the grid as a constraint and formulate the topology identification problem as a blind learning problem, which can estimate the graph Laplacian accurately. All the claims have been verified in simulations using IEEE test case data with the 14-bus system as an example.

II. System Models

Power systems are characterized by *buses* that represent interconnections, generators or loads, denoted by the set $\mathcal{N} \triangleq$

This work was supported by the U.S. Department of Energy through the Trustworthy Cyber Infrastructure for Power Grid (TCIPG) program.

$\{1, \cdots, N\}$. The wiring between buses is determined by the edge set $\mathcal{E} \triangleq \{\{n, m\}\}$ with cardinality $|\mathcal{E}| = E$, which corresponds to the wiring between bus n and m. Each line is characterized by the line impedance $Z_{nm} = R_{nm} + \mathrm{i}X_{nm}$, $\{n, m\} \in \mathcal{E}$ in the Π-model of line $\{n, m\} \in \mathcal{E}$.

A. Power Systems as Graphs

A power system can be effectively described as an undirected graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ with the node set $\mathcal{N}$ and edge set $\mathcal{E}$. The associated topology can be further described by the *oriented incidence matrix* consisting of columns $\mathbf{m}_{nm} \in \{-1, 0, 1\}^{N \times 1}$ for each edge $\{n, m\} \in \mathcal{E}$ defined as follows

$$[\mathbf{m}_{nm}]_i = \begin{cases} 1, & i = n \\ -1, & i = m \\ 0, & \text{otherwise} \end{cases}. \tag{1}$$

Accordingly, the $N \times E$ incidence matrix of $\mathcal{G}$ is given by

$$\mathbf{M} \triangleq [\cdots, \mathbf{m}_{nm}, \cdots], \quad n < m \text{ and } \{n, m\} \in \mathcal{E}, \tag{2}$$

Using the incidence matrix, the *weighted Laplacian* of the graph is a symmetric matrix defined as

$$\mathbf{L} \triangleq \mathbf{M}\mathbf{W}\mathbf{M}^T, \tag{3}$$

where $\mathbf{W} = \mathrm{diag}[\cdots, W_{nm}, \cdots]$ is a diagonal weight matrix containing individual weights W_{nm} for each edge in the set $\{n, m\} \in \mathcal{E}$. Note that by definition the Laplacian is positive semi-definite $\mathbf{L} \succeq \mathbf{0}$ and has the following *null space property*:

$$\mathbf{L}\mathbf{1} = \mathbf{0}, \quad \mathbf{L}^T\mathbf{1} = \mathbf{0}. \tag{4}$$

In power grids, the weight on each edge is tied to the physical parameter of the line, namely the electrical admittance of each line $W_{nm} = Z_{nm}^{-1}$, $\{n, m\} \in \mathcal{E}$. Therefore, the negative of the weighted Laplacian $-\mathbf{L}$ is often called the admittance matrix[1], and the electrical properties of the system become deterministic once $\mathbf{L}$ is given.

B. Data Model

Here we consider a limited data situation in which no direct information on the grid topology is given, and the only available data are the power injections P_n at each bus n. For simplicity, we consider the linear DC power flow model over a period $t = 1, \cdots, T$,

$$\mathbf{p}_t = \mathbf{B}\boldsymbol{\theta}_t + \mathbf{r}_t, \quad t = 1, \cdots, T, \tag{5}$$

where $\mathbf{p}_t \triangleq [P_1(t), \cdots, P_N(t)]^T$, $\boldsymbol{\theta}_t \triangleq [\theta_1(t), \cdots, \theta_N(t)]^T$ and $\mathbf{r}_t = [r_1(t), \cdots, r_N(t)]^T$ correspond respectively to the power injections, voltage phase angles and measurement noise at each time t. The matrix $\mathbf{B}$ is the load flow matrix given by

$$\mathbf{B} \triangleq -\mathbf{M}\mathbf{X}^{-1}\mathbf{M}^T \tag{6}$$

with $\mathbf{X} = \mathrm{diag}[\cdots, X_{nm}, \cdots]$ by ignoring all the resistances $R_{nm} \approx 0$. Note that the DC model in (5) is not invertible due to the null space property

$$\mathbf{B}\mathbf{1} = \mathbf{0}, \quad \mathbf{B}^T\mathbf{1} = \mathbf{0}, \tag{7}$$

and hence it is common practice to choose a reference bus angle (i.e., the slack bus) to resolve the ambiguity in $\boldsymbol{\Theta}$:

$$\boldsymbol{\Theta}^T\mathbf{1}_{\mathrm{ref}} = \boldsymbol{\theta}_{\mathrm{ref}} \tag{8}$$

where $\mathbf{1}_{\mathrm{ref}} = [1, 0, \cdots, 0]^T$ and usually $\boldsymbol{\theta}_{\mathrm{ref}} = \mathbf{0}$.

By stacking the T snapshots of injection data $\{\mathbf{p}_t\}_{t=1}^T$ in an $N \times T$ data matrix $\mathbf{P} = [\mathbf{p}_1, \cdots, \mathbf{p}_T]$, the block data model in this paper is given by

$$\mathbf{P} = \mathbf{B}\boldsymbol{\Theta} + \mathbf{R}, \tag{9}$$

where $\boldsymbol{\Theta} = [\boldsymbol{\theta}_1, \cdots, \boldsymbol{\theta}_T]$ is the phase angle matrix and $\mathbf{R} = [\mathbf{r}_1, \cdots, \mathbf{r}_N]$ is the noise matrix. Furthermore, we impose the following condition on the input $\boldsymbol{\Theta}$ phase angles.

Condition 1. *(Sufficient Excitation) The phase angle matrix $\boldsymbol{\Theta} \in \mathbb{R}^{N \times T}$ is full rank such that $T \geq N$ and $\mathrm{rank}(\boldsymbol{\Theta}) = N$.*

This condition can be met by taking measurements with sufficient angle variations over time. Furthermore, for simplicity, we assume that the noise samples are independent and identically distributed (i.i.d.) zero mean Gaussian random variables, and hence the noise covariance becomes $\mathbb{E}\{\mathbf{r}_t\mathbf{r}_\tau^T\} = \sigma^2\mathbf{I}\delta[t - \tau]$ for $t, \tau = 1, \cdots, T$.

III. Blind Topology Identification

A. Topology Identifiability

In this paper, we aim to identify the topology of the grid by learning the matrix $\mathbf{B}$ based solely on the injection data $\mathbf{P}$ without any knowledge of $\boldsymbol{\Theta}$. In other words, apart from certain structures of $\mathbf{B}$ implied by the null space property in (4) and the negative semi-definiteness $\mathbf{B} \preceq \mathbf{0}$ by definition in (6), this is a completely blind estimation problem with respect to $\mathbf{B}$. Clearly, this is a challenging task which may not be solvable. In the following, we first prove an important result regarding the identifiability of the topology.

Theorem 1. *Given the Laplacian matrix $\mathbf{B}$ of a general graph $\mathcal{G} = \{\mathcal{N}, \mathcal{E}\}$, then under the null space property (7) and $\mathbf{B} \preceq \mathbf{0}$, the matrix $\mathbf{B}$ cannot be uniquely unidentified using solely injection data $\mathbf{P}$ in (9) satisfying Condition 1.*

Proof: See Appendix A. ∎

Although it is impossible to recover the exact Laplacian matrix $\mathbf{B}$ using only power injection data, we prove in the following theorem that the identified Laplacian matrix $\widetilde{\mathbf{B}}$ has the same eigenvectors as the true Laplacian $\mathbf{B}$.

Theorem 2. *Given the Laplacian matrix $\mathbf{B}$ of a general graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ and its eigenvalue decomposition $\mathbf{B} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{U}^T$, then under the null space property (7) and $\mathbf{B} \preceq \mathbf{0}$, if the injection data $\mathbf{P}$ in (9) are used, the Laplacian matrix $\widetilde{\mathbf{B}} = \mathbf{U}\widetilde{\boldsymbol{\Sigma}}\mathbf{U}^T$ is identifiable up to a scale in the eigenvalues $\widetilde{\boldsymbol{\Sigma}} = \mathbf{D}\boldsymbol{\Sigma}$ (i.e., $\mathbf{D}$ is an arbitrary positive definite diagonal matrix).*

Proof: See Appendix B. ∎

It is asserted in Theorem 2 that the matrix $\mathbf{B}$ is always ambiguous up to a diagonal scaling in the eigenvalues, which is non-unique. Therefore, to better refine the estimate, other constraints are needed. An important feature of the power grid that can aid the identification is the sparsity of the graph $\mathcal{G}$, as

[1]Note that the admittance matrix also contains bus admittance-to-ground, which is neglected here for convenience.

studied in [24] and [25]. Given a graph $\mathcal{G}$ with $|\mathcal{E}| = E$ edges, we have the following sparsity constraint on $\mathbf{B}$ below:

$$\|\text{vec}(\mathbf{B})\|_0 \leq 2E. \tag{10}$$

However, imposing the sparsity constraints forces to solve complex combinatorial searches. Therefore we relax the requirement by choosing a sparsity-enhancing prior on the elements in $\mathbf{B}$, which is the Laplace distribution

$$\mathbb{P}(\mathbf{B}) = \frac{1}{2b} \exp\left(-\frac{1}{b}\|\text{vec}(\mathbf{B})\|_1\right) \tag{11}$$

with sparsity parameter $b > 0$.

B. Problem Formulation as Sparse Subspace Learning

By assuming the voltage phase angles $\boldsymbol{\Theta}$ to be deterministic unknowns, the unknown $\mathbf{B}$ and $\boldsymbol{\Theta}$ can be jointly estimated in the *maximum a posteriori* (MAP) framework by maximizing the posterior probability

$$\mathbb{P}(\mathbf{B}|\mathbf{P}; \boldsymbol{\Theta}) \propto \mathbb{P}(\mathbf{P}|\mathbf{B}; \boldsymbol{\Theta})\mathbb{P}(\mathbf{B}) \tag{12}$$

subject to the null space property in (7), the negative semi-definiteness $\mathbf{B} \preceq \mathbf{0}$, and the slack reference of $\boldsymbol{\Theta}$ in (8). Hence, under the Gaussian noise assumption and the sparsity prior in (11), the constrained MAP estimator is formulated as

$$\begin{aligned}\left\{\widehat{\mathbf{B}}, \widehat{\boldsymbol{\Theta}}\right\} = \min_{\mathbf{B},\boldsymbol{\Theta}} \; & \|\mathbf{P} - \mathbf{B}\boldsymbol{\Theta}\|_F^2 + \mu\|\text{vec}(\mathbf{B})\|_1 \\ \text{s.t.} \;\; & \boldsymbol{\Theta}^T \mathbf{1}_{\text{ref}} = \boldsymbol{\theta}_{\text{ref}} \\ & \mathbf{B}\mathbf{1} = \mathbf{B}^T\mathbf{1} = \mathbf{0}, \;\; \mathbf{B}^T = \mathbf{B}, \;\; \mathbf{B} \preceq \mathbf{0},\end{aligned} \tag{13}$$

where $\mu = \sigma^2/b$ is the sparsity regularization parameter. Note that the sparsity regularization $\|\text{vec}(\mathbf{B})\|_1$ helps further resolve the ambiguity of the estimates $\{\widehat{\mathbf{B}}, \widehat{\boldsymbol{\Theta}}\}$ from (13) by constraining the eigenvalue scaling $\widetilde{\boldsymbol{\Sigma}}$.

The topology identification problem in (13) bears a certain resemblance to dictionary learning [26] with major distinctions. Typically, in such problems the excitation $\boldsymbol{\Theta}$ is given by a set of sparse vectors that are projected onto an overcomplete dictionary $\mathbf{B}$. The matrix $\mathbf{B}$ is normally non-sparse and represents what is often called the dictionary (or sensing matrix in the compressed sensing literature). In our problem, instead, the system $\mathbf{B}$ is sparse, and it has additional structural constraints (7). In contrast, the excitation matrix $\boldsymbol{\Theta}$ is generally overcomplete and closer to playing the role of the *unknown dictionary*. The above issues make our problem more challenging. Also, it is to be noted that (13) is non-convex, and it is computationally prohibitive to solve for both $\mathbf{B}$ and $\boldsymbol{\Theta}$ jointly. Therefore, following the rationale of alternating projections as commonly used in dictionary learning [26], we switch between estimating $\boldsymbol{\Theta}$ and $\mathbf{B}$ to update the input signals $\boldsymbol{\Theta}$ and the sparse subspace $\mathbf{B}$ iteratively.

C. Learning by Block Coordinate Descent (BCD)

Similar to the block coordinate descent approach used for dictionary learning [26], we propose to decouple the problem in (13) in a two-step manner with some initial guess on $\mathbf{B}_0$:

1) Phase updates:

$$\boldsymbol{\Theta}_k = \min_{\boldsymbol{\Theta}} \|\mathbf{P} - \mathbf{B}_k\boldsymbol{\Theta}\|_F^2, \quad \text{s.t. } \boldsymbol{\Theta}^T\mathbf{1}_{\text{ref}} = \boldsymbol{\theta}_{\text{ref}}.$$

Algorithm 1 Learning Algorithm for Topology Identification

obtain data $\mathbf{P}$ in (9)
set $k = 0$ and initialize $\mathbf{B}_0$;
while $k \leq K_{\max}$ **do**
  initialize $\mathbf{B}_k^{(0)} = \mathbf{B}_{k-1}$
  repeat
    the i-th subgradient descents (15) and (16);
  until $i \leq I_{\max}$ or $\|\boldsymbol{\Theta}_k^{(i)} - \boldsymbol{\Theta}_k^{(i-1)}\|_F \leq \epsilon$
  obtain $\boldsymbol{\Theta}_k = \boldsymbol{\Theta}_k^{(i)}$;
  repeat
    the i-th subgradient descents (20), (21) and (22);
  until $i \leq I_{\max}$ or $\|\mathbf{B}_k^{(i)} - \mathbf{B}_k^{(i-1)}\|_F \leq \epsilon$
  obtain $\mathbf{B}_k = \mathbf{B}_k^{(i)}$;
  if $\|\mathbf{B}_k - \mathbf{B}_{k-1}\| \leq \epsilon$ **then**
    terminate **while** loop
  end if
  $k = k + 1$;
end while

2) Topology updates:

$$\begin{aligned}\mathbf{B}_k = \min_{\mathbf{B}} \; & \|\mathbf{P} - \mathbf{B}\boldsymbol{\Theta}_k\|_F^2 + \mu\|\text{vec}(\mathbf{B})\|_1 \\ \text{s.t.} \;\; & \mathbf{B}\mathbf{1} = \mathbf{B}^T\mathbf{1} = \mathbf{0}, \;\; \mathbf{B}^T = \mathbf{B}, \;\; \mathbf{B} \preceq \mathbf{0}.\end{aligned}$$

In this way, each update becomes a constrained convex optimization problem, which can be solved efficiently by primal-dual subgradient methods [27] specified below.

1) Phase Updates: Introducing the dual variables $\boldsymbol{\xi} \in \mathbb{R}_+^{T\times 1}$ for the slack reference constraint for $\boldsymbol{\Theta}$, the Lagrangian function of the k-th phase update optimization is obtained as

$$\mathcal{L}_k(\boldsymbol{\Theta}, \boldsymbol{\xi}) = \|\mathbf{P} - \mathbf{B}_k\boldsymbol{\Theta}\|_F^2 + \boldsymbol{\xi}^T(\boldsymbol{\Theta}^T\mathbf{1}_{\text{ref}} - \boldsymbol{\theta}_{\text{ref}}). \tag{14}$$

Then starting with an initial $\boldsymbol{\Theta}_k^{(0)}$ and $\boldsymbol{\xi}_k^{(0)}$, the i-th primal-dual subgradient updates for the k-th phase update are written as

$$\boldsymbol{\Theta}_k^{(i+1)} = \boldsymbol{\Theta}_k^{(i)} - \alpha_i \frac{\partial}{\partial \boldsymbol{\Theta}} \mathcal{L}_k(\boldsymbol{\Theta}_k^{(i)}, \boldsymbol{\xi}_k^{(i)}) \tag{15}$$

$$\boldsymbol{\xi}_k^{(i+1)} = \left[\boldsymbol{\xi}_k^{(i)} - \alpha_i \frac{\partial}{\partial \boldsymbol{\xi}} \mathcal{L}_k(\boldsymbol{\Theta}_k^{(i)}, \boldsymbol{\xi}_k^{(i)})\right]^+, \tag{16}$$

where the step-size satisfies $\sum_{i=1}^{\infty} \alpha_i = \infty$ and $\sum_{i=1}^{\infty} \alpha_i^2 < \infty$, and the subgradients are given as follows:

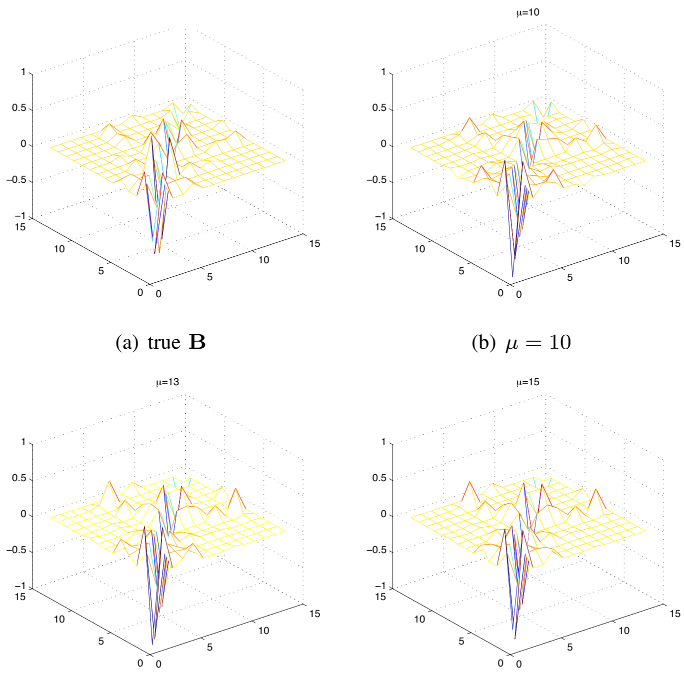
$$\frac{\partial}{\partial \boldsymbol{\Theta}} \mathcal{L}_k(\boldsymbol{\Theta}, \boldsymbol{\xi}) = 2\mathbf{B}_k^T(\mathbf{B}_k\boldsymbol{\Theta} - \mathbf{P}) + \mathbf{1}_{\text{ref}}\boldsymbol{\xi}^T \tag{17}$$

$$\frac{\partial}{\partial \boldsymbol{\xi}} \mathcal{L}_k(\boldsymbol{\Theta}, \boldsymbol{\xi}) = \boldsymbol{\Theta}^T\mathbf{1}_{\text{ref}} - \boldsymbol{\theta}_{\text{ref}}. \tag{18}$$

2) Topology Updates: Similarly, introducing dual variables $\boldsymbol{\rho} \in \mathbb{R}_+^{N\times 1}$ and $\boldsymbol{\Lambda} \in \mathbb{R}_+^{N\times N}$, the Lagrangian function of the k-th topology update optimization is written as

$$\begin{aligned}\mathcal{L}_k(\mathbf{B}, \boldsymbol{\Lambda}, \boldsymbol{\rho}) = & \|\mathbf{P} - \mathbf{B}\boldsymbol{\Theta}_k\|_F^2 + \mu\|\text{vec}(\mathbf{B})\|_1 \\ & + \boldsymbol{\rho}^T(\mathbf{B}\mathbf{1} + \mathbf{B}^T\mathbf{1}) + \text{Tr}\left[\boldsymbol{\Lambda}(\mathbf{B}^T - \mathbf{B})\right]\end{aligned} \tag{19}$$

over the constraint set $\mathcal{B} \triangleq \{\mathbf{B} : \; \mathbf{B} \preceq \mathbf{0}\}$. For convenience, we define the projection operator $\mathcal{P}_{\mathcal{B}}[\cdot]$ onto the subspace $\mathcal{B}$ by retaining the eigenvectors of a certain matrix corresponding to the non-zero eigenvalues.



(a) true $\mathbf{B}$ (b) $\mu = 10$ (c) $\mu = 13$ (d) $\mu = 15$

Fig. 1. Recovered $\widehat{\mathbf{B}}$ and the true $\mathbf{B}$

Given the initial points $\boldsymbol{\rho}_k^{(0)}$ and $\boldsymbol{\Lambda}_k^{(0)}$, the updates for the k-th topology update optimization are thus obtained as

$$\mathbf{B}_k^{(i+1)} = \mathcal{P}_{\mathcal{B}}\left[\mathbf{B}_k^{(i)} - \beta_i \frac{\partial}{\partial \mathbf{B}} \mathcal{L}_k(\mathbf{B}_k^{(i)}, \boldsymbol{\Lambda}_k^{(i)}, \boldsymbol{\rho}_k^{(i)})\right] \tag{20}$$

$$\boldsymbol{\Lambda}_k^{(i+1)} = \left[\boldsymbol{\Lambda}_k^{(i)} - \beta_i \frac{\partial}{\partial \boldsymbol{\Lambda}} \mathcal{L}_k(\mathbf{B}_k^{(i)}, \boldsymbol{\Lambda}_k^{(i)}, \boldsymbol{\rho}_k^{(i)})\right]^+ \tag{21}$$

$$\boldsymbol{\rho}_k^{(i+1)} = \left[\boldsymbol{\rho}_k^{(i)} - \beta_i \frac{\partial}{\partial \boldsymbol{\rho}} \mathcal{L}_k(\mathbf{B}_k^{(i)}, \boldsymbol{\Lambda}_k^{(i)}, \boldsymbol{\rho}_k^{(i)})\right]^+, \tag{22}$$

where the step-size satisfies $\sum_{i=1}^{\infty} \beta_i = \infty$ and $\sum_{i=1}^{\infty} \beta_i^2 < \infty$. The subgradients are given by

$$\frac{\partial}{\partial \mathbf{B}} \mathcal{L}_k(\mathbf{B}, \boldsymbol{\Lambda}, \boldsymbol{\rho}) = 2(\mathbf{B}\boldsymbol{\Theta}_k - \mathbf{P})\boldsymbol{\Theta}_k^T + (\boldsymbol{\Lambda} - \boldsymbol{\Lambda}^T) \tag{23}$$

$$+ (\boldsymbol{\rho}\mathbf{1}^T + \mathbf{1}\boldsymbol{\rho}^T) + \mu \mathcal{F}(\mathbf{B}) \tag{24}$$

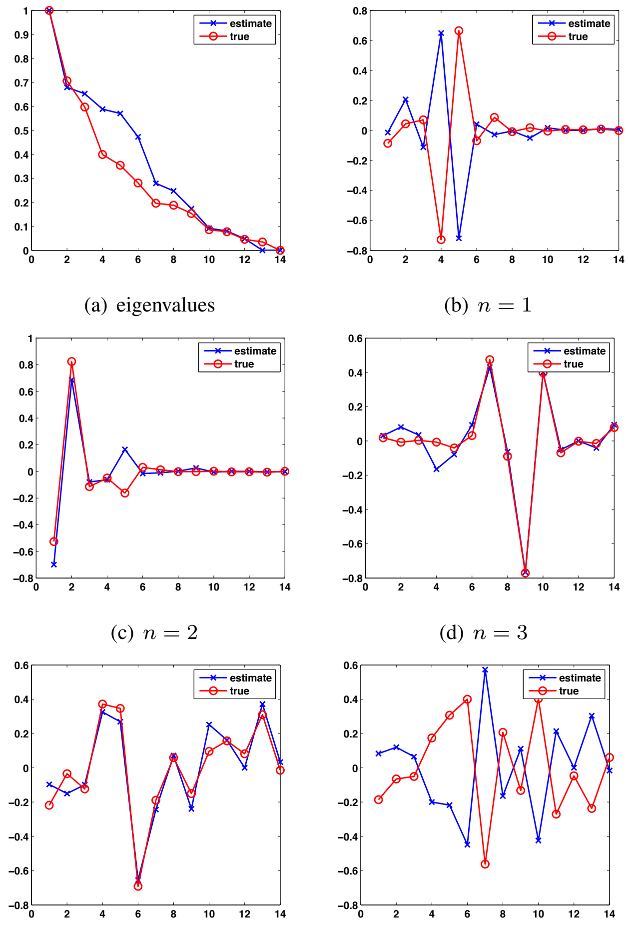
$$\frac{\partial}{\partial \boldsymbol{\Lambda}} \mathcal{L}_k(\mathbf{B}, \boldsymbol{\Lambda}, \boldsymbol{\rho}) = \mathbf{B} - \mathbf{B}^T \tag{25}$$

$$\frac{\partial}{\partial \boldsymbol{\rho}} \mathcal{L}_k(\mathbf{B}, \boldsymbol{\Lambda}, \boldsymbol{\rho}) = \mathbf{B}\mathbf{1} + \mathbf{B}^T\mathbf{1}, \tag{26}$$

where $\mathcal{F}(\mathbf{B}) = \partial \|\mathrm{vec}(\mathbf{B})\|_1 / \partial \mathbf{B}$ is the subgradient of ℓ_1 norm with respect to $\mathbf{B}$.

IV. Simulations

In this section, we illustrate the identification performance of the grid topology using our proposed method. In the simulations, we use the MATPOWER 4.0 test case IEEE-14 ($N = 14$) system. We take the load profile from the UK National Grid [28] and scale the base load from MATPOWER on load buses. Then the Optimal Power Flow (OPF) determines the generation dispatch over this period using an AC model. This gives us the true state $\boldsymbol{\theta}_t$ and the injection data $\mathbf{p}_t$ for $t = 1, \cdots, T$, with $T = 100$ snapshots. To implement our algorithm in Algorithm 1, we choose $\alpha_i = \beta_i = 0.1/i$ for the phase and topology updates using the primal-dual subgradient method with $\epsilon = 10^{-6}$, $I_{\max} = 100$ and $K_{\max} = 100$.



(a) eigenvalues (b) $n = 1$ (c) $n = 2$ (d) $n = 3$ (e) $n = 4$ (f) $n = 5$

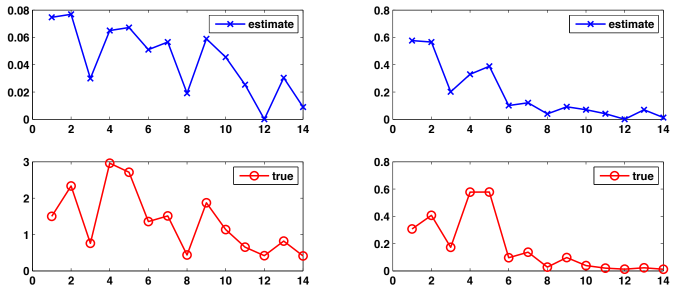
Fig. 2. Principal eigenvectors of the identified matrix $\widehat{\mathbf{B}}$ and the true $\mathbf{B}$

A. Recovery Performance

Since the weighted Laplacian matrix cannot be perfectly recovered, the first point of interest is to examine how well it captures the connectivity $\mathcal{E}$ and the relative weights B_{nm} on the links $\{n, m\} \in \mathcal{E}$ of the underlying graph. In Fig. 1, we plot the matrix $\widehat{\mathbf{B}}$ using sparsity regularization $\mu = 10, 13$ and 15 respectively, as two- dimensional functions with the elements $\widehat{B}_{nm}$ as the intensity at location $\{n, m\}$. It can be seen from the sparsity pattern and the image intensity that the graph Laplacians can be identified quite accurately except for the scale, which demonstrates that our algorithm can effectively learn the grid topology by simply taking power injection data.

B. Subspace Identification Performance

Although the identified Laplacian $\widehat{\mathbf{B}}$ is always ambiguous with $\mathbf{B}$ in the eigenvalue scaling, the eigenvectors are uniquely identifiable. Therefore, Fig. 2 demonstrates the identification performance of our learning algorithm by comparing the



(a) $C_d(n)$ (b) $C_v(n)$

Fig. 3. Centrality measures of the recovered graph associated with $\widehat{\mathbf{B}}$ and the true graph associated with $\mathbf{B}$

principal eigenvectors $\{\mathbf{u}_n\}_{n=1}^{5}$ of $\widehat{\mathbf{B}}$ and $\mathbf{B}$. Based on Fig. 1, only the results obtained with $\mu = 13$ are presented here while other cases are omitted because of similar performances.

The reason for showing the principal components is mainly due to the fact that grid Laplacian has been shown in [25] to have rapidly decaying eigenvalues and sparse eigenvectors, and also due to space limitation. Therefore, if the principal components are identified correctly, critical information on the topology of the graph can be obtained accordingly. It can be seen from Figs. 2(b) to 2(f) that our method is effective and efficient in capturing the subspace of the graph since the estimated eigenvectors almost overlap with the true eigenvectors $\{\pm\mathbf{u}_n\}_{n=1}^{5}$. Last but not least, we also compare the normalized eigenvalues of both matrices in Fig. 2(a), which have been rescaled by the respective maximum eigenvalue. It is clear that the recovered normalized eigenvalues follow a decaying pattern similar to that of the true graph.

C. Properties of the Identified Graph

Apart from the graph pattern and the subspace identification performance, it is also interesting to examine how the recovered graph retains some important graph properties. As pointed out in [24], [25] and [15], a critical measure in determining the risk of having cascading failures is the *centrality* of the grid [15]. In particular, we examine the weighted *degree centrality*

$$C_d(n) = \frac{|B_{nn}|}{N-1} \tag{27}$$

and the weighted *eigenvector centrality* of the grid

$$C_u(n) = \frac{1}{\max(\sigma_A^{(n)})}\mathbf{A}_{(n,:)}\left[\mathbf{U}_A\right]_{(:,1)}, \tag{28}$$

which is the n-th entry of the most significant eigenvector of the weighted adjacency matrix $\mathbf{A} \triangleq \mathbf{B} - \mathrm{diag}(\mathbf{B})$ with eigenvalue decomposition $\mathbf{A} = \mathbf{U}_A\boldsymbol{\Sigma}_A\mathbf{U}_A^T$. The degree centrality represents the connectivity of a node to the rest of the network and reflects the immediate chance for a node to exert its influences on the rest of the network or to be exposed to whatever is flowing through the network, while the eigenvector centrality is a measure of the importance of a node in a network according to its adjacency matrix.

In Fig. 3, we plot the centrality measures for the recovered graph and the true graph. It can be seen that although there is a scale difference, the relative scale on the centrality measures are retained satisfactorily. Therefore, by acquiring the injection data, we can identify the critical nodes in the power grid for security purposes.

V. Conclusions

In this paper, we have proposed a learning algorithm for blind topology identification of power grids. Rigorous analysis has been presented on the identifiability of grid topology, and the performance of our algorithm has been well illustrated in simulations.

Appendix A
Proof of Theorem 1

We prove our claim by contradiction. Assume that the topology $\mathbf{B}$ can be uniquely identified from $\mathbf{P}$. This implies that for any $\mathbf{B} \neq \widetilde{\mathbf{B}}$, the following always holds for any $\widetilde{\boldsymbol{\Theta}}$:

$$\mathbf{B}\boldsymbol{\Theta} \neq \widetilde{\mathbf{B}}\widetilde{\boldsymbol{\Theta}}, \tag{29}$$

which leads to

$$\begin{bmatrix}\mathbf{B} & \widetilde{\mathbf{B}}\end{bmatrix}\begin{bmatrix}\boldsymbol{\Theta} \\ -\widetilde{\boldsymbol{\Theta}}\end{bmatrix} \neq \mathbf{0}. \tag{30}$$

This is true for any $\widetilde{\boldsymbol{\Theta}}$ only if the null space dimension of the matrix $[\mathbf{B}\ \widetilde{\mathbf{B}}]$ is less than the range space dimension of the matrix $[\boldsymbol{\Theta}^T \ -\widetilde{\boldsymbol{\Theta}}^T]^T$. It is well known that

$$\nu_B \triangleq \dim\ker\,[\mathbf{B}\ \widetilde{\mathbf{B}}] = 2N - \mathrm{rank}[\mathbf{B}\ \widetilde{\mathbf{B}}] \tag{31}$$

and for any $\widetilde{\boldsymbol{\Theta}}$, from Condition 1 there is

$$\nu_\Theta \triangleq \inf_{\widetilde{\boldsymbol{\Theta}}} \mathrm{rank}\begin{bmatrix}\boldsymbol{\Theta} \\ -\widetilde{\boldsymbol{\Theta}}\end{bmatrix} = N, \tag{32}$$

and therefore by the assumption imposed in (30) we have

$$\nu_B < \nu_\Theta \Longrightarrow \mathrm{rank}[\mathbf{B}\ \widetilde{\mathbf{B}}] > 2N - \nu_\Theta \geq N. \tag{33}$$

Now, since both $\mathbf{B}$ and $\widetilde{\mathbf{B}}$ are both graph Laplacians and satisfy the null space property in (7) and the negative semi-definiteness of $\mathbf{B}$, it is clear that

$$\mathrm{span}(\mathbf{B}), \mathrm{span}(\widetilde{\mathbf{B}}) \subseteq \mathrm{span}\left(\mathbf{I}_N - \mathbf{1}\mathbf{1}^T/N\right) \tag{34}$$

which results in

$$\mathrm{rank}[\mathbf{B}\ \widetilde{\mathbf{B}}] \leq N-1. \tag{35}$$

This clearly contradicts the result $\mathrm{rank}[\mathbf{B}\ \widetilde{\mathbf{B}}] > N$ in (33) derived from (30). Therefore, with only the null space property in (7) and negative semi-definiteness, it is impossible to recover $\mathbf{B}$ uniquely from the injection data $\mathbf{P}$.

Appendix B
Proof of Theorem 2

Suppose there exists another distinct pair $\widetilde{\mathbf{B}} \neq \mathbf{B}$ and $\widetilde{\boldsymbol{\Theta}} \neq \boldsymbol{\Theta}$ that satisfies $\mathbf{P} = \widetilde{\mathbf{B}}\widetilde{\boldsymbol{\Theta}}$ and therefore

$$\mathbf{B}\boldsymbol{\Theta} = \widetilde{\mathbf{B}}\widetilde{\boldsymbol{\Theta}}. \tag{36}$$

By Condition 1, the matrix $\boldsymbol{\Theta}$ is of full rank and therefore the right pseudo-inverse exists, which gives

$$\mathbf{B} = \tilde{\mathbf{B}}\widetilde{\boldsymbol{\Theta}}\boldsymbol{\Theta}^T\left(\boldsymbol{\Theta}\boldsymbol{\Theta}^T\right)^{-1}. \tag{37}$$

By right multiplying $\mathbf{\Theta}$ on both sides, we have from (36) that

$$\mathbf{B\Theta} = \tilde{\mathbf{B}}\widetilde{\mathbf{\Theta}}\mathbf{\Theta}^T \left(\mathbf{\Theta\Theta}^T\right)^{-1}\mathbf{\Theta} = \tilde{\mathbf{B}}\widetilde{\mathbf{\Theta}}. \tag{38}$$

The above equality on the right hand side is equivalent to

$$\tilde{\mathbf{B}}\left[\widetilde{\mathbf{\Theta}} - \widetilde{\mathbf{\Theta}}\mathbf{\Theta}^T\left(\mathbf{\Theta\Theta}^T\right)^{-1}\mathbf{\Theta}\right] = \mathbf{0}. \tag{39}$$

From the null space property (7) of $\widetilde{\mathbf{B}}$, we can conclude that

$$\widetilde{\mathbf{\Theta}}\left[\mathbf{I} - \mathbf{\Theta}^T\left(\mathbf{\Theta\Theta}^T\right)^{-1}\mathbf{\Theta}\right] = \mathbf{1c}^T, \tag{40}$$

where $\mathbf{c}$ is an arbitrary constant vector that is not in the row space of $\mathbf{\Theta}$ (i.e., $\mathbf{c} \notin \mathrm{span}(\mathbf{\Theta}^T)$). It is to be noted that the right hand side of $\widetilde{\mathbf{\Theta}}$ is a projection onto the orthogonal subspace to the row space of $\mathbf{\Theta}$. Therefore, it is clear that the row span of $\mathbf{\Theta}$ is orthogonal to $\mathbf{1c}^T$ and there exists a non-singular linear transform $\mathbf{T}$ that relates $\mathbf{\Theta}$ and $\widetilde{\mathbf{\Theta}}$ as

$$\widetilde{\mathbf{\Theta}} = \mathbf{T\Theta} + \mathbf{1c}^T. \tag{41}$$

Substituting (41) back into (36), we have

$$\mathbf{B\Theta} = \widetilde{\mathbf{B}}\left(\mathbf{T\Theta} + \mathbf{1c}^T\right) = \widetilde{\mathbf{B}}\mathbf{T\Theta}, \tag{42}$$

which leads to

$$\left(\mathbf{B} - \widetilde{\mathbf{B}}\mathbf{T}\right)\mathbf{\Theta} = \mathbf{0}. \tag{43}$$

Since $\mathbf{\Theta}$ is of full rank by Condition 1, the equivalent condition for (36) to hold is to have

$$\mathbf{B} = \widetilde{\mathbf{B}}\mathbf{T}. \tag{44}$$

By definition, $\mathbf{B}$ is symmetric and satisfies the null space property (7), and hence the ambiguous solution of $\widetilde{\mathbf{B}}$ needs to further satisfy

$$\widetilde{\mathbf{B}}\mathbf{T} = (\widetilde{\mathbf{B}}\mathbf{T})^T = \mathbf{T}^T\widetilde{\mathbf{B}}^T, \quad \mathbf{T1} \propto \mathbf{1}, \quad \mathbf{T}^T\mathbf{1} \propto \mathbf{1}. \tag{45}$$

A sufficient condition for the above equalities to hold is to have $\mathbf{T}$ jointly diagonalizable with $\mathbf{B} = \mathbf{U\Sigma U}^T$ using $\mathbf{U}$

$$\mathbf{T} = \mathbf{UD}^{-1}\mathbf{U}^T, \tag{46}$$

where $\mathbf{D} = \mathrm{diag}[d_1, \cdots, d_N]$ with $d_n > 0$ for $n = 1, \cdots, N$. Therefore, the ambiguous solution $\widetilde{\mathbf{B}}$ has a special structure

$$\widetilde{\mathbf{B}} = \mathbf{U}\widetilde{\mathbf{\Sigma}}\mathbf{U}^T, \tag{47}$$

where $\widetilde{\mathbf{\Sigma}} = \mathbf{D\Sigma}$.

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